

# $S$ -matrix for $s$ -wave gravitational scattering

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## Abstract

In the  $s$ -wave approximation the 4D Einstein gravity with scalar fields can be reduced to an effective 2D dilaton gravity coupled nonminimally to the matter fields. We study the leading order (tree level) vertices. The 4-particle matrix element is calculated explicitly. It is interpreted as scattering with formation of a virtual black hole state. As one novel feature we predict the gravitational decay of  $s$ -waves.

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## 1 Introduction

Quantum gravity is beset with well-known conceptual problems. Probably the most challenging one is the dual rôle of geometric variables as fields which at the same time determine the local and global properties of the manifold on which they act. Due to this fact and because gravity is perturbatively non-renormalizable it is desirable to use non-perturbative methods. Unfortunately, in  $d = 4$  this is technically problematic.

Therefore, models in  $d = 2$  are considered frequently in this context, most of which lack an important feature present in ordinary gravity: They contain no continuous physical degrees of freedom. One way to overcome this without leaving the comfortable realm of two dimensions is the inclusion of matter.

The aim of the present work is to shed some light on a 2d system which is closely related to Einstein gravity in  $d = 4$  and thus of some phenomenological relevance, namely the spherically reduced Einstein-massless-Klein-Gordon model. It exhibits many interesting properties already at the classical level. Our exact treatment of the geometric part allows for the straightforward calculation of the non-local vertex, which is interpreted as the exchange of a virtual black hole. The (highly non-trivial) classical  $S$ -matrix resulting from this graph is determined and discussed in this paper. Furthermore our approach provides the basis for quantum corrections in the matter sector.

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## 2 First order formulation

When the line element\*

$$(ds)_{(4)}^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta - X(x)d\Omega_{S^2}^2 \quad (1)$$

only depends on the metric on the unit 2-sphere,  $d\Omega_{S^2}^2$ , the “dilaton field”  $X$  and a two dimensional metric  $g_{\alpha\beta}$  with signature  $(+, -)$ , the Hilbert-Einstein action, supplemented by an action in which the scalar field  $S$  is coupled minimally (in  $d = 4$ ) to gravity, becomes equivalent to a two dimensional dilaton theory [1]

$$L_{\text{dil}} = \frac{4\pi}{\kappa} \int d^2x \sqrt{-g} \left[ XR + \frac{(\nabla X)^2}{2X} - 2 + \kappa X (\nabla S)^2 \right]. \quad (2)$$

$R$  denotes the  $2d$  curvature,  $(\nabla X)^2 = g^{\alpha\beta} \partial_\alpha X \partial_\beta X$ , and  $\kappa = 8\pi G_N$ .

The action (2) is locally and globally equivalent to a first order one [2] depending on Cartan 1-forms  $e^\pm$  (we denote light-cone indices with  $\pm$ ) and  $\omega$  (the abelian gauge structure of the two dimensional spin connection  $\omega^a{}_b = \varepsilon^a{}_b \omega$  is used explicitly), the dilaton field  $X$ , the auxiliary fields  $X^\pm$  and the  $s$ -wave Klein-Gordon field  $S$ :

$$\begin{aligned} L_{\text{FO}} = & -\frac{8\pi}{\kappa} \int [X^+(d - \omega) \wedge e^- + X^-(d + \omega) \wedge e^+ + \\ & + X d \wedge \omega - e^- \wedge e^+ \mathcal{V} - \frac{\kappa}{2} X dS \wedge *dS]. \end{aligned} \quad (3)$$

Actually, this equivalence holds for general dilaton theories. In the present spherically reduced case the “potential” in (3) becomes  $\mathcal{V} = -2 - \frac{X^+ X^-}{2X}$ . In the following we shall take  $\kappa = 1$  and drop the overall factor. Only in the final result the full  $\kappa$ -dependence will be restored.

## 3 Path integral quantization of geometry

Although in the present paper we consider only classical (tree level) processes, the path integral seems to be the most adequate language to derive the scattering amplitudes. As in other well-known examples (e.g. the Klein-Nishina formula for relativistic Compton scattering [3]) this formalism is much superior to a purely classical computation, because it directly focuses on the physical observable, the  $S$ -matrix element, which leads to an immediate interpretation. In a series of papers [4–6] it has been shown that the path integral quantization, developed from the action (3), allows the exact treatment of the geometric part for the choice of a temporal gauge of the Cartan variables

$$\omega_0 = 0, \quad e_0^- = 1, \quad e_0^+ = 0. \quad (4)$$

Previous work was restricted to minimally coupled scalars in (3), i.e. the dilaton factor  $X$  in front of the matter action was omitted.

The Hamiltonian analysis in terms of the remaining field variables and associated conjugate momenta

$$q_i = (\omega_1, e_1^-, e_1^+), \quad p_i = (X, X^+, X^-), \quad (5)$$

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\* $x := (x^0, x^1)$  and the indices  $\alpha, \beta$  go from 0 to 1

together with the introduction of the path integral in phase space, suitably extended by ghosts works here as in [4–6]. Though, as a consequence of the dilaton factor  $X$  in (3), the structure functions of the constraint algebra acquire additional terms, but the nilpotent BRST charge also here resembles the one in Yang-Mills theories. For details of the Hamiltonian analysis and path integral quantization we refer to [7]. Having integrated ghost fields and other canonical variables, the effective action including sources for  $q_i, p_i$  and  $S$  differs only slightly from the one in [5, 6]:

$$L = \int \left[ -\dot{p}_i q_i + q_1 p_2 - q_3 \mathcal{V} + \frac{p_1}{2} (\partial_1 S \partial_0 S - q_2 (\partial_0 S)^2) + j_i q_i + J_i p_i + Q S \right]. \quad (6)$$

The generating functional for the Green functions reads

$$Z[j, J, Q] = \int (\mathcal{D}q) (\mathcal{D}p) (\mathcal{D}S) \exp(iL). \quad (7)$$

After the (exact)  $q$ - and  $p$ -integrals only the integration of scalars remains. Thus, the usual perturbation theory is restricted to the incorporation of matter fields. Separating terms of  $\mathcal{O}(S^{2n})$ ,  $n > 2$ , the Gaussian path integral of the terms up to  $\mathcal{O}(S^2)$  yields a typical propagator contribution, apart from terms of  $\mathcal{O}(\hbar)$ , like a generalized Polyakov action and a contribution from the measure. As in ref. [5] we concentrate on the (highly nontrivial) vertex  $\mathcal{O}(S^4)$  in the perturbation expansion. It allows the calculation of scattering of  $s$ -wave scalars. This vertex can be extracted formally from the final effective action. However, it contains complicated multiple integrals. Hence, we use again the simple short cut introduced in [5], the idea of which we will outline briefly: It is sufficient to assume the second order combinations of the scalar field to be localized at a single point<sup>†</sup>

$$S_0 := \frac{1}{2} (\partial_0 S)^2 = c_0 \delta(x - y), \quad (8)$$

$$S_1 := \frac{1}{2} (\partial_0 S) (\partial_1 S) = c_1 \delta(x - y), \quad (9)$$

and to solve the classical equations of motion (EOM) following from the gauge fixed action (6) up to linear order in the “sources”  $c_0$  or  $c_1$ . Then the solutions have to be substituted back into the interaction terms in (6). Higher orders in  $c_0, c_1$  would yield either loop contributions or vertices with at least 6 outer legs. We emphasize again that we are using perturbative methods in the matter sector only. Thus no a priori split into background- and fluctuation-metric occurs in our approach.

## 4 Classical EOM

The solution of the classical EOM in the presence of matter from (6) with vanishing sources

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<sup>†</sup> Actually, the sources should be localized at different points, but for the lowest order tree graphs – which are our main goal – this makes no difference.

$$\begin{aligned}
\partial_0 p_1 &= p_2, & \partial_0 q_1 &= \frac{q_3 p_2 p_3}{2p_1^2} + S_1 - q_2 S_0, \\
\partial_0 p_2 &= p_1 S_0, & \partial_0 q_2 &= -q_1 - \frac{q_3 p_3}{2p_1}, \\
\partial_0 p_3 &= 2 + \frac{p_2 p_3}{2p_1}, & \partial_0 q_3 &= -\frac{q_3 p_2}{2p_1},
\end{aligned} \tag{10}$$

to linear order in  $c_0$  and  $c_1$  is found easily:

$$p_1(x) = x_0 + (x_0 - y_0)c_0 y_0 h(x, y), \tag{11}$$

$$p_2(x) = 1 + c_0 y_0 h(x, y), \tag{12}$$

$$\begin{aligned}
q_2(x) &= 4\sqrt{p_1} + \left( 2c_0 y_0^{3/2} - c_1 y_0 \right. \\
&\quad \left. + (c_1 - 6c_0 y_0^{1/2})p_1 \right) h(x, y),
\end{aligned} \tag{13}$$

$$q_3(x) = \frac{1}{\sqrt{p_1}}. \tag{14}$$

Here  $h(x, y) := \theta(y_0 - x_0)\delta(x_1 - y_1)$ , corresponds to one of the prescriptions introduced in [6] for the boundary values at  $x_0 \rightarrow \infty$ . It turns out that the vertices below are *independent of any* such choice. The matching conditions at  $x_0 = y_0$  follow from continuity properties:  $p_1, q_2$  and  $q_3$  are  $C^0$  and  $\partial_0 q_2(y_0 + 0) - \partial_0 q_2(y_0 - 0) = (c_1 - q_2(y_0)c_0)\delta(x_1 - y_1)$ . Integration constants which would produce an asymptotic (i.e. for  $x_0 \rightarrow \infty$ ) Schwarzschild term and a Rindler term have been fixed to zero. Thus, a black hole may appear only at an intermediate stage (the “virtual black hole”, see below), but should not act asymptotically. Due to the infinite range of gravity this is necessary for a proper  $S$ -matrix element, if we want to use as asymptotic states spherical waves for the incoming and outgoing scalar particles.

## 5 Line element

The matter dependent solutions in our gauge (4) from (5), (13) and (14) define an effective line element

$$(ds)^2 = 2drdu + K(r, u)(du)^2, \tag{15}$$

if we identify<sup>‡</sup>  $u = 2\sqrt{2}x_1$  and  $r = \sqrt{p_1(x_0)}/2$ . It then appears in outgoing Sachs-Bondi form. The Killing norm

$$K(r, u)|_{x_0 < y_0} = \left( 1 - \frac{2m}{r} - ar \right) (1 + \mathcal{O}(c_0)), \tag{16}$$

with  $m = \delta(x_1 - y_1)(-c_1 y_0 - 2c_0 y_0^{3/2})/2^{7/2}$  and  $a = \delta(x_1 - y_1)(c_1 - 6c_0 y_0^{1/2})/2^{5/2}$ , has two zeros located approximately at  $r = 2m$  and  $r = 1/a$  corresponding for positive  $m$  and  $a$  to a Schwarzschild horizon and a Rindler type one. In the asymptotic region the Killing norm is constant by fixing  $K(r, u)|_{x_0 > y_0} = 1$ .

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<sup>‡</sup>Note the somewhat unusual rôle of the indices 0 and 1:  $x_0$  is asymptotically proportional to  $r^2$ , thus our Hamiltonian evolution is with respect to a “radius” as “time”-parameter.

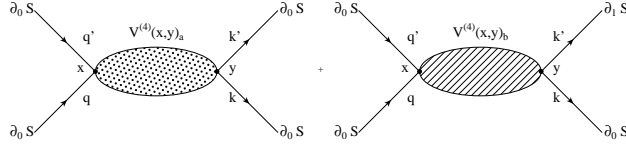


Figure 1: Total  $V^{(4)}$ -vertex with outer legs

## 6 Virtual black hole (VBH)

As in [6] we turn next to the conserved quantity, which exists in all two dimensional generalized dilaton theories [8], even in the presence of matter [9,10]. For SRG its geometric part reads

$$\mathcal{C}^{(g)} = \frac{p_2 p_3}{\sqrt{p_1}} - 4\sqrt{p_1} \quad (17)$$

and by assumption it vanishes in the asymptotic region  $x_0 > y_0$ . A simple argument shows that  $\mathcal{C}^{(g)}$  is discontinuous:  $p_1$  and  $p_3$  are continuous, but  $p_2$  jumps at  $x_0 = y_0$ . This phenomenon has been called “virtual black hole” (VBH) in [6]. It is generic rather than an artifact of our special choice of asymptotic conditions. The reason why we have chosen this name is simple: The geometric part of the conserved quantity (17) is essentially equivalent to the so-called mass aspect function, which is closely related to the black hole mass [10]. Moreover, inspection of the Killing norm (16) reveals, that for very small Rindler acceleration  $a$  the Schwarzschild horizon corresponds to a BH with precisely that mass. This BH disappears in the asymptotic states (by construction), but mediates an interaction between them.

The idea that black holes must be considered in the  $S$ -matrix together with elementary matter fields has been put forward some time ago [11]. Our approach has allowed for the first time to derive (rather than suppose) the existence of the black hole states in the quantum scattering matrix. So far, we were able to perform actual computations in the first non-trivial order only. The next order calculations which should yield an insight into the information paradox are in progress.

The solutions (11) and (12) establish

$$\mathcal{C}^{(g)} \Big|_{x_0 < y_0} = 4c_0 y_0^{3/2} \propto -m_{VBH}. \quad (18)$$

Thus,  $c_1$  only enters the Rindler term in the Killing norm, but not the VBH mass (18).

## 7 The $S^4$ vertex

All integration constants have been fixed by the arguments in the preceding paragraphs. The fourth order vertex of quantum field theory is extracted by collecting the terms quadratic in  $c_0$  and  $c_1$  replacing each by  $S_0$  and  $S_1$ , respectively. The tree graphs we obtain in that way (cf. fig. 1) contain the nonlocal

vertices

$$\begin{aligned}
V_a^{(4)} &= \int_x \int_y S_0(x) S_0(y) \left( \frac{dq_2}{dc_0} p_1 + q_2 \frac{dp_1}{dc_0} \right) \Big|_{c_i=0} \\
&= \int_x \int_y S_0(x) S_0(y) |\sqrt{y_0} - \sqrt{x_0}| \sqrt{x_0 y_0} \\
&\quad (3x_0 + 3y_0 + 2\sqrt{x_0 y_0}) \delta(x_1 - y_1),
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
V_b^{(4)} &= \int_x \int_y \left( S_0(y) S_1(x) \frac{dp_1}{dc_0} - S_0(x) S_1(y) \frac{dq_2}{dc_1} p_1 \right) \Big|_{c_i=0} \\
&= \int_x \int_y S_0(x) S_1(y) |x_0 - y_0| x_0 \delta(x_1 - y_1),
\end{aligned} \tag{20}$$

with  $\int_x := \int_0^\infty dx^0 \int_{-\infty}^\infty dx^1$ .

## 8 Asymptotics

With  $t := r + u$  the scalar field satisfies asymptotically the spherical wave equation. For proper  $s$ -waves only the spherical Bessel function

$$R_{k0}(r) = \frac{\sin(kr)}{kr} \tag{21}$$

survives in the mode decomposition ( $Dk := 4\pi k^2 dk$ ):

$$S(r, t) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty \frac{Dk}{\sqrt{2k}} R_{k0} [a_k^+ e^{ikt} + a_k^- e^{-ikt}]. \tag{22}$$

With  $a^\pm$  obeying the commutation relation  $[a_k^-, a_{k'}^+] = \delta(k - k')/(4\pi k^2)$ , they will be used to define asymptotic states and to build the Fock space. The normalization factor is chosen such that the Hamiltonian reads

$$H = \frac{1}{2} \int_0^\infty Dr [(\partial_t S)^2 + (\partial_r S)^2] = \int_0^\infty Dk a_k^+ a_k^- k. \tag{23}$$

## 9 Scattering amplitude

In [6] we had arrived at a trivial result in the massless case for (in  $d = 2$ ) minimally coupled scalars: Either the  $S$ -matrix was divergent or – if the VBH was “plugged” by suitable boundary conditions on  $S$  at  $r = 0$  – it vanished. Only for massive scalars we found some finite nonvanishing scattering amplitude.

In the present physical case of  $s$ -waves from  $d = 4$  General Relativity at a first glance it may be surprising that the simple additional factor  $X$  in front of the matter Lagrangian induces fundamental changes in the qualitative behavior. In fact, it causes the partial differential equations (10) to become coupled, giving rise to an additional vertex ( $V_b^{(4)}$ ).

After a long and tedious calculation (for details see [7, 12]) for the  $S$ -matrix element with ingoing modes  $q, q'$  and outgoing ones  $k, k'$

$$T(q, q'; k, k') = \frac{1}{2} \left\langle 0 \left| a_k^- a_{k'}^- \left( V_a^{(4)} + V_b^{(4)} \right) a_q^+ a_{q'}^+ \right| 0 \right\rangle \quad (24)$$

having restored the full  $\kappa$ -dependence we arrive at

$$T(q, q'; k, k') = -\frac{i\kappa\delta(k+k'-q-q')}{2(4\pi)^4 |kk'qq'|^{3/2}} E^3 \tilde{T} \quad (25)$$

with  $E = q + q'$ ,

$$\begin{aligned} \tilde{T}(q, q'; k, k') := & \frac{1}{E^3} \left[ \Pi \ln \frac{\Pi^2}{E^6} + \frac{1}{\Pi} \sum_{p \in \{k, k', q, q'\}} p^2 \ln \frac{p^2}{E^2} \right. \\ & \left. \cdot \left( 3kk'qq' - \frac{1}{2} \sum_{r \neq p} \sum_{s \neq r, p} (r^2 s^2) \right) \right], \end{aligned} \quad (26)$$

and  $\Pi = (k+k')(k-q)(k'-q)$ . The interesting part of the scattering amplitude is encoded in the scale independent factor  $\tilde{T}$ .

## 10 Discussion

The simplicity of (26) is quite surprising, in view of the fact that the two individual contributions (cf. figure 1) are not only vastly more complicated, but also divergent. This precise cancellation urgently asks for some deeper explanation. The fact that a particular subset of graphs to a given order in perturbation theory may be gauge dependent and even divergent, while the sum over all such subsets should yield some finite, gauge-independent  $S$ -matrix is well known from gauge theory in particle physics (cf. e.g. [13]). However, it seems that only in the temporal gauge (4) one is able to integrate out the geometric degrees of freedom successfully. Also that gauge is free from coordinate singularities which we believe to be a prerequisite for a dynamical study extending across the horizon<sup>§</sup>.

The only possible singularities occur if an outgoing momentum equals an ingoing one (forward scattering). Near such a pole we obtain with  $k = q + \varepsilon$  and  $q \neq q'$ :

$$\tilde{T}(q, q'; \varepsilon) = \frac{2(qq')^2}{\varepsilon} \ln \left( \frac{q}{q'} \right) + \mathcal{O}(1). \quad (27)$$

The nonlocality of the vertex prevents the calculation of the usual  $s$ -wave cross section. However, an analogous quantity can be defined by squaring (25) and dividing by the spacetime integral over the product of the densities of the incoming waves ( $\rho = (2\pi)^{-3} \sin^2(qr)/(qr)^2$ ):  $I = \int Dr dt \rho(q) \rho(q')$ ,  $\sigma = I^{-1} \int_0^\infty Dk Dk' |T|^2$ . Together with the introduction of dimensionless kinematic variables  $k = E\alpha$ ,  $k' = E(1-\alpha)$ ,  $q = E\beta$ ,  $q' = E(1-\beta)$ ,  $\alpha, \beta \in [0, 1]$  this yields

$$\frac{d\sigma}{d\alpha} = \frac{1}{4(4\pi)^3} \frac{\kappa^2 E^2 |\tilde{T}(\alpha, \beta)|^2}{(1-|\beta-1|)(1-\alpha)(1-\beta)\alpha\beta}. \quad (28)$$

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<sup>§</sup>Other gauges of this class, e.g. the Painlevé-Gullstrand gauge [14] seem to be too complicated to allow an application of our present approach.

Our result also allows the definition of a decay rate  $d^3\Gamma/(DqDkDk')$  of an  $s$ -wave with ingoing momentum  $q$  decaying (!) into three outgoing ones with momenta  $k, k', -q'$ . Clearly, lifetimes calculated in this manner will crucially depend on assumed distributions for the momenta.

Finally, we stress that in the more general four dimensional setup of gravitational particle scattering combinations of non-spherical modes could contribute to the  $s$ -wave matrix element. Hence, our result (25) does not include the full ( $4d$ ) classical information. Nonetheless, as the previous discussion shows, its physical content is highly nontrivial. We emphasize especially the decay of  $s$ -waves, which is a new phenomenon caused by the non-linearity of the underlying theory. Note that it is not triggered by graviton interaction, since there are no spherically symmetric gravitons. Still, it is caused by gravity, i.e. by gravitational self interaction encoded in our non local vertices. Though the existence of such processes may be expected on general grounds, our simple method allows us to calculate the corresponding amplitudes explicitly.

Our methods are useful also for other applications, such as spherically symmetric collapse [15] or the polarized Gowdy model [16].

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